FOURIER TRANSFORMS

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The Fourier series gives a series representation of a periodic function. The *Fourier transform* generalizes this idea to a function defined over an infinite range, that need not be periodic. In effect, we can think of the function has having an infinite period.

Saff and Snider give a heuristic derivation of the Fourier transform in Section 8.2. The results are that the Fourier transform of a function $F(t)$ is given by

$$G(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(t)\,e^{-i\omega t}dt \tag{1}$$

with the inverse transform given by

$$F(t)=\int_{-\infty}^{\infty}G(\omega)\,e^{i\omega t}d\omega \tag{2}$$

Note that the location of the 2π varies among various books and computer programs. For example, Maple puts the $\frac{1}{2\pi}$ in front of the integral for the inverse transform and omits it in the definition of the original transform.

Fourier transforms can be calculated exactly only for a restricted set of functions $F(t)$ due to the difficulty of doing the integrals. Here, we'll give a few examples of cases where $G(\omega)$ can actually be calculated exactly.

Example 1. Find the Fourier transform of

$$F(t)=e^{-|t|} \tag{3}$$

We can split this into two integrals as

$$G(\omega)=\frac{1}{2\pi}\int_{-\infty}^{0}e^{t-i\omega t}dt+\frac{1}{2\pi}\int_{0}^{\infty}e^{-t-i\omega t}dt \tag{4}$$

$$=-\frac{1}{2\pi(i\omega-1)}+\frac{1}{2\pi(i\omega+1)} \tag{5}$$

$$=\frac{1+i\omega}{2\pi(\omega^2+1)}+\frac{1-i\omega}{2\pi(\omega^2+1)} \tag{6}$$

$$=\frac{1}{\pi(\omega^2+1)} \tag{7}$$

Example 2. Find the Fourier transform of

$$F(t)=e^{-t^2} \tag{8}$$

We have

$$G_2(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-t^2-i\omega t}dt \tag{9}$$

We can complete the square in the exponent to get

$$-t^2-i\omega t=-\left(t+\frac{i\omega}{2}\right)^2-\frac{\omega^2}{4} \tag{10}$$

so we have

$$G_2(\omega)=\frac{1}{2\pi}e^{-\omega^2/4}\int_{-\infty}^{\infty}e^{-(t+i\omega/2)^2}dt \tag{11}$$

We can use the substitution

$$u=t+\frac{i\omega}{2} \tag{12}$$

The limits on the integral remain the same, from $u=-\infty$ to $u=\infty$, and $dt=du$, so we have

$$G_2(\omega)=\frac{1}{2\pi}e^{-\omega^2/4}\int_{-\infty}^{\infty}e^{-u^2}du \tag{13}$$

The integral is a Gaussian integral, with the value

$$\int_{-\infty}^{\infty}e^{-u^2}du=\sqrt{\pi} \tag{14}$$

so the Fourier transform is

$$G_2(\omega)=\frac{1}{2\pi}e^{-\omega^2/4}\sqrt{\pi}=\frac{1}{2\sqrt{\pi}}e^{-\omega^2/4} \tag{15}$$

Example 3. Find the Fourier transform of

$$F(t)=te^{-t^2} \tag{16}$$

This time, we're faced with the integral

$$G_3(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}te^{-t^2-i\omega t}dt \tag{17}$$

Writing this as

$$G_3(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[te^{-t^2}\right]\left[e^{-i\omega t}\right]dt$$

we can integrate by parts to get

$$G_3(\omega) = \frac{1}{2\pi}\frac{e^{-t^2}}{-2}\,e^{-i\omega t}\Big|_{-\infty}^{\infty} - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-i\omega}{-2}e^{-t^2-i\omega t}dt \tag{18}$$

$$= 0-0-\frac{i\omega}{4\pi}\int_{-\infty}^{\infty}e^{-t^2-i\omega t}dt \tag{19}$$

$$= -\frac{i\omega}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-t^2-i\omega t}dt\right] \tag{20}$$

The remaining integral in the square brackets is the same as in Example 2, so the final transform is

$$G_3(\omega) = -\frac{i\omega}{2}\frac{1}{2\sqrt{\pi}}e^{-\omega^2/4} = -\frac{i\omega}{4\sqrt{\pi}}e^{-\omega^2/4} \tag{21}$$

In this case, the transform is purely imaginary.

There's actually a faster way of finding the transform here. Taking the derivative of 9 with respect to ω (assuming we can differentiate inside the integral), we have

$$\frac{dG_2(\omega)}{d\omega} = -\frac{i}{2\pi}\int_{-\infty}^{\infty}te^{-t^2-i\omega t}dt = -iG_3(\omega) \tag{22}$$

From 15 we have

$$\frac{dG_2(\omega)}{d\omega} = -\frac{\omega}{4\sqrt{\pi}}e^{-\omega^2/4} = -iG_3(\omega) \tag{23}$$

so

$$G_3(\omega) = -\frac{i\omega}{4\sqrt{\pi}}e^{-\omega^2/4} \tag{24}$$

which agrees with 21.

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